Proofs: Supplementary Material for paper ‘Enhanced Secondary Frequency Control via Distributed Peer-to-Peer Communication’

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1 Condition for Ignoring Ramping Constraints

Under Assumption 3, Theorem 2 shows that if \( \|\Delta P_L(t) - \Delta P_L(t-1)\| \leq \epsilon, \forall t \in \mathcal{T}, \) there exists a \( c, \) such that \( \|\lambda^t_n - \lambda^t\| \leq c\epsilon. \) Assumption 3 further guarantees:

\[
\|u_i(t+1) - u_i(t)\| \leq \beta \sum_{i \in \Omega} \|\lambda_i(t) - \lambda_i(t)\| + \frac{1}{n}\|\Delta P_L(t) - \Delta P_L(t-1)\|
\]

\[
\leq \beta \sum_{i \in \Omega} (\|\lambda_i(t) - \lambda(t)\| + \|\lambda(t) - \lambda_i(t)\|) + \frac{1}{n}\epsilon
\]

\[
\leq \left(2\beta c|\Omega_i| + \frac{1}{n}\right) \epsilon.
\]

Thus, as long as

\[
\left(2\beta c|\Omega_i| + \frac{1}{n}\right) \epsilon \leq r_i, \forall i \in \Omega,
\]

where \( r_i \) is regulation resource \( i \)'s ramping limit, no ramping constraints will be binding. Hence, in such conditions, our relaxation is exact.

2 Proof of Theorem 1

**Theorem 1:** The proposed distributed control scheme satisfies the necessary condition (5).

**Proof:** At each time \( t \in \mathcal{T}, \) a step load change may happen at the beginning of the time slot, and according to the update rule, we have

\[
\sum_{i \in \Omega} u_i(t+1) = \sum_{i \in \Omega} \frac{\hat{\lambda}_i^{t+1}}{2a_i}, \quad (sp-1)
\]

and

\[
\sum_{i \in \Omega} \frac{\hat{\lambda}_i^{t+1}}{2a_i} = \sum_{i \in \Omega} \frac{\hat{\lambda}_i^t}{2a_i} - \beta \sum_{i \in \Omega} \sum_{t \in \mathcal{T}} (\lambda^t_i - \lambda^t_j) + \Delta P_L(t+1) - \sum_{i \in \Omega} \Delta P_m^i(t). \quad (sp-2)
\]
Note the following facts hold:

\[
\sum_{i \in \Omega} \sum_{l \in \Omega} (\lambda^t_i - \lambda^t_l) = 0, \forall t \in T; \quad (\text{sp-3})
\]

\[
\sum_{i \in \Omega} \tilde{\lambda}^t_i = \sum_{i \in \Omega} \Delta P^t_m(t). \quad (\text{sp-4})
\]

Combining (sp-3) and (sp-4) with (sp-2), we establish that

\[
\sum_{i \in \Omega} u_i(t + 1) = \Delta P_L(t + 1). \quad (\text{sp-5})
\]

\[\blacksquare\]

## 3 Proof of Theorem 2

**Theorem 2** If the communication graph is well connected such that (28) holds, \(\|\Delta P_L(t + 1) - \Delta P_L(t)\| < \epsilon, \forall t \in T\), and Assumption 3 holds, then there exists a constant \(c > 0\) (depending only on the cost parameters and the communication topology) such that control signals \((u_i)'s\) are \(c\epsilon\)-close to their optimal values given by (13)-(14). More precisely,

\[
\|\lambda^t_i - \lambda^t\| \leq c\epsilon, \forall t \in T, i \in \Omega,
\]

\[
\|u_i(t) - u^*_i(t)\| \leq c\epsilon, \forall t \in T, i \in \Omega,
\]

where \(u^*_i(t)\) denotes the solution to (13)-(14).

**Proof:** Using Assumption 3, we know that \(\lambda^t = \tilde{\lambda}^t\), for all \(t \in T\). Thus, the marginal cost updates may be written in vector notation as

\[
A\lambda_{t+1} = (A - \beta L)\lambda_t + n^{-1}(\Delta P_L(t + 1) - \Delta P_L(t))1. \quad (\text{sp-6})
\]

Since \(1^T L = 0^T\), we have

\[
1^T A\lambda_{t+1} = 1^T (A - \beta L)\lambda_t + (\Delta P_L(t + 1) - \Delta P_L(t)). \quad (\text{sp-7})
\]

Assumption 3 also leads to

\[
1^T A\lambda_t = 1^T A1\lambda^t. \quad (\text{sp-8})
\]

Combining (sp-8) with (sp-7) yields

\[
1^T A1\lambda^{t+1} = 1^T A1\lambda^t + (\Delta P_L(t + 1) - \Delta P_L(t)). \quad (\text{sp-9})
\]

By dividing \(1^T A1\) at both sides, we have

\[
\lambda^{t+1} = \lambda^t + \frac{\Delta P_L(t + 1) - \Delta P_L(t)}{1^T A1}. \quad (\text{sp-10})
\]
Therefore,
\[ \lambda_{t+1} - \lambda^t \mathbf{1} \]
\[ = (I - \beta \Lambda^{-1} L) \lambda_t + (n\Lambda)^{-1} (\Delta P_L(t+1) - \Delta P_L(t)) \mathbf{1} - \lambda^t \mathbf{1} - \frac{\Delta P_L(t+1) - \Delta P_L(t)}{1^T \Lambda \mathbf{1}} \]
\[ = (I - \beta \Lambda^{-1} L) (\lambda_t - \lambda^t \mathbf{1}) + (n\Lambda)^{-1} (\Delta P_L(t+1) - \Delta P_L(t)) \mathbf{1} \]
\[ \leq (I - \beta \Lambda^{-1} L) (\lambda_t - \lambda^t \mathbf{1}) + \left\| (n\Lambda)^{-1} - \frac{1}{1^T \Lambda \mathbf{1}} \right\| \epsilon \mathbf{1}. \] (sp-11)

Note that in the second equation of (sp-11), we add a zero term \( \beta \Lambda^{-1} L \lambda^t \mathbf{1} \). By contradiction, we can show that the largest eigenvalue of the matrix \( I - \beta \Lambda^{-1} L \) is 1. Note that,
\[ I - \beta \Lambda^{-1} L = \Lambda^{-\frac{1}{2}} (I - \beta \Lambda^{-\frac{1}{2}} L \Lambda^{-\frac{1}{2}}) \Lambda^{\frac{1}{2}}, \] (sp-12)
\( I - \beta \Lambda^{-1} L \) and \( I - \beta \Lambda^{-\frac{1}{2}} L \Lambda^{-\frac{1}{2}} \) are similar matrices with the same eigenvalues. Since the Laplacian \( L \) corresponds to a connected network, one is a simple eigenvalue of the matrix \( I - \beta \Lambda^{-1/2} L \Lambda^{-1/2} \) with corresponding eigenvector \( \Lambda^{\frac{1}{2}} \mathbf{1} \). By (sp-8), we have
\[ (\Lambda^{\frac{1}{2}} \mathbf{1})^T \Lambda^{\frac{1}{2}} (\lambda_t - \lambda^t \mathbf{1}) = 1^T \Lambda (\lambda_t - \lambda^t \mathbf{1}) = 0. \] (sp-13)

Thus, \( \Lambda^{\frac{1}{2}} (\lambda_t - \lambda^t \mathbf{1}) \) is orthogonal to the eigenvector corresponding to the eigenvalue 1. Furthermore, the matrix \( \Lambda^{-1/2} L \Lambda^{-1/2} \) is positive semidefinite and hence, by taking \( \beta \) to be small enough, all other eigenvalues of \( I - \beta \Lambda^{-1/2} L \Lambda^{-1/2} \) can be guaranteed to lie in the interval \( [0,1) \). Since \( I - \beta \Lambda^{-\frac{1}{2}} L \Lambda^{-\frac{1}{2}} \) is a symmetric matrix, the induced 2-norm coincides with the spectral radius, thus
\[ \|(I - \Lambda^{-1} \beta L)(\lambda_t - \lambda^t \mathbf{1})\| \]
\[ = \|\Lambda^{-\frac{1}{2}} (I - \beta \Lambda^{-\frac{1}{2}} L \Lambda^{-\frac{1}{2}}) \Lambda^{\frac{1}{2}} (\lambda_t - \lambda^t \mathbf{1})\| \]
\[ \leq \|\Lambda^{-\frac{1}{2}}\| \|I - \beta \Lambda^{-\frac{1}{2}} L \Lambda^{-\frac{1}{2}}\| \|\Lambda^{\frac{1}{2}}\| \|\lambda_t - \lambda^t \mathbf{1}\| \]
\[ \leq (1 - \rho) \max_{i \in \mathcal{N}} a_i^{\frac{1}{2}} \max_{i \in \mathcal{N}} (a_i)^{-\frac{1}{2}} \|\lambda_t - \lambda^t \mathbf{1}\|, \] (sp-14)

where \( 1 - \rho \in [0,1) \) is the second largest eigenvalue of \( I - \beta \Lambda^{-1} L \). As long as (28) holds, standard algebraic manipulations now lead to the desired conclusion. ■