Proofs: Supplementary Material for paper 'Enhanced Secondary Frequency Control via Distributed Peer-to-Peer Communication'

Chenye Wu, Soummya Kar and Gabriela Hug

1 Condition for Ignoring Ramping Constraints

Under Assumption 3, Theorem 2 shows that if $\|\Delta P_L(t) - \Delta P_L(t-1)\| \le \epsilon,\ \forall t \in \mathcal{T}$, there exists a c, such that $\|\lambda_n^t - \lambda^t\| \le c\epsilon$. Assumption 3 further guarantees:

$$\begin{aligned}\|u_i(t+1) - u_i(t)\| \le& \beta \sum_{l\in\Omega_i} \|\lambda_i(t) - \lambda_l(t)\| + \frac{1}{n}\|\Delta P_L(t) - \Delta P_L(t-1)\| \\ \le& \beta \sum_{l\in\Omega_i} (\|\lambda_i(t) - \lambda(t)\| + \|\lambda(t) - \lambda_l(t)\|) + \frac{1}{n}\epsilon \\ \le& \left(2\beta c|\Omega_i| + \frac{1}{n}\right)\epsilon.\end{aligned}$$

Thus, as long as

$$\left(2\beta c|\Omega_i| + \frac{1}{n}\right)\epsilon \le r_i, \forall i \in \Omega,$$

where r_i is regulation resource i's ramping limit, no ramping constraints will be binding. Hence, in such conditions, our relaxation is exact.

2 Proof of Theorem 1

Theorem 1: *The proposed distributed control scheme satisfies the necessary condition (5).*

Proof: At each time $t \in \mathcal{T}$, a step load change may happen at the beginning of the time slot, and according to the update rule, we have

$$\sum_{i\in\Omega} u_i(t+1) = \sum_{i\in\Omega} \frac{\tilde{\lambda}_i^{t+1}}{2a_i}, \tag{sp-1}$$

and

$$\sum_{i\in\Omega} \frac{\tilde{\lambda}_i^{t+1}}{2a_i} = \sum_{i\in\Omega} \frac{\tilde{\lambda}_i^{t}}{2a_i} - \beta \sum_{i\in\Omega}\sum_{l\in\Omega_i} (\lambda_i^t - \lambda_l^t) + \Delta P_L(t+1) - \sum_{i\in\Omega} \Delta P_m^i(t). \tag{sp-2}$$

Note the following facts hold:

$$\sum_{i\in\Omega}\sum_{l\in\Omega_i}(\lambda_i^t-\lambda_l^t)=0, \forall t\in\mathcal{T}; \tag{sp-3}$$

$$\sum_{i\in\Omega}\frac{\tilde{\lambda}_i^t}{2a_i}=\sum_{i\in\Omega}\Delta P_m^i(t). \tag{sp-4}$$

Combining (sp-3) and (sp-4) with (sp-2), we establish that

$$\textstyle\sum_{i\in\Omega}u_i(t+1)=\Delta P_L(t+1). \tag{sp-5}$$

■

3 Proof of Theorem 2

Theorem 2 *If the communication graph is well connected such that (28) holds,* $\|\Delta P_L(t+1)-\Delta P_L(t)\|<\epsilon$, $\forall t\in\mathcal{T}$, *and Assumption 3 holds, then there exists a constant* $c>0$ *(depending only on the cost parameters and the communication topology) such that control signals (u_i's) are $c\epsilon$-close to their optimal values given by (13)-(14). More precisely,*

$$\|\lambda_i^t-\lambda^t\|\le c\epsilon,\ \forall t\in\mathcal{T}, i\in\Omega,$$
$$\|u_i(t)-u_i^*(t)\|\le c\epsilon,\ \forall t\in\mathcal{T}, i\in\Omega,$$

where $u_i^*(t)$ denotes the solution to (13)-(14).

Proof: Using Assumption 3, we know that $\lambda^t=\tilde{\lambda}^t$, for all $t\in\mathcal{T}$. Thus, the marginal cost updates may be written in vector notation as

$$\Lambda\boldsymbol{\lambda}_{t+1}=(\Lambda-\beta L)\boldsymbol{\lambda}_t+n^{-1}(\Delta P_L(t+1)-\Delta P_L(t))\mathbf{1}. \tag{sp-6}$$

Since $\mathbf{1}^TL=0^T$, we have

$$\mathbf{1}^T\Lambda\boldsymbol{\lambda}_{t+1}=\mathbf{1}^T\Lambda\boldsymbol{\lambda}_t+(\Delta P_L(t+1)-\Delta P_L(t)). \tag{sp-7}$$

Assumption 3 also leads to

$$\mathbf{1}^T\Lambda\boldsymbol{\lambda}_t=\mathbf{1}^T\Lambda\mathbf{1}\lambda^t. \tag{sp-8}$$

Combining (sp-8) with (sp-7) yields

$$\mathbf{1}^T\Lambda\mathbf{1}\lambda^{t+1}=\mathbf{1}^T\Lambda\mathbf{1}\lambda^t+(\Delta P_L(t+1)-\Delta P_L(t)). \tag{sp-9}$$

By dividing $\mathbf{1}^T\Lambda\mathbf{1}$ at both sides, we have

$$\lambda^{t+1}=\lambda^t+\frac{\Delta P_L(t+1)-\Delta P_L(t)}{\mathbf{1}^T\Lambda\mathbf{1}}. \tag{sp-10}$$

Therefore,

$$\begin{aligned}
&\boldsymbol{\lambda}_{t+1} - \lambda^{t+1}\mathbf{1} \\
=&(I - \beta\Lambda^{-1}L)\boldsymbol{\lambda}_t + (n\Lambda)^{-1}(\Delta P_L(t+1) - \Delta P_L(t))\mathbf{1} - \lambda^t\mathbf{1} - \frac{\Delta P_L(t+1) - \Delta P_L(t)}{\mathbf{1}^T\Lambda\mathbf{1}}\mathbf{1} \\
=&(I - \beta\Lambda^{-1}L)(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}) + \left((n\Lambda)^{-1} - \frac{1}{\mathbf{1}^T\Lambda\mathbf{1}}\right)(\Delta P_L(t+1) - \Delta P_L(t))\mathbf{1} \\
\leq&(I - \beta\Lambda^{-1}L)(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}) + \left\|(n\Lambda)^{-1} - \frac{1}{\mathbf{1}^T\Lambda\mathbf{1}}\right\|\epsilon\mathbf{1}.
\end{aligned} \tag{sp-11}$$

Note that in the second equation of (sp-11), we add a zero term $\beta\Lambda^{-1}L\lambda^t\mathbf{1}$. By contradiction, we can show that the largest eigenvalue of the matrix $I - \beta\Lambda^{-1}L$ is 1. Note that,

$$I - \beta\Lambda^{-1}L = \Lambda^{-\frac{1}{2}}(I - \beta\Lambda^{-\frac{1}{2}}L\Lambda^{-\frac{1}{2}})\Lambda^{\frac{1}{2}}, \tag{sp-12}$$

$I - \beta\Lambda^{-1}L$ and $I - \beta\Lambda^{-\frac{1}{2}}L\Lambda^{-\frac{1}{2}}$ are similar matrices with the same eigenvalues. Since the Laplacian L corresponds to a connected network, one is a simple eigenvalue of the matrix $I - \beta\Lambda^{-1/2}L\Lambda^{-1/2}$ with corresponding eigenvector $\Lambda^{\frac{1}{2}}\mathbf{1}$. By (sp-8), we have

$$(\Lambda^{\frac{1}{2}}\mathbf{1})^T\Lambda^{\frac{1}{2}}(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}) = 1^T\Lambda(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}) = 0. \tag{sp-13}$$

Thus, $\Lambda^{\frac{1}{2}}(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1})$ is orthogonal to the eigenvector corresponding to the eigenvalue 1. Furthermore, the matrix $\Lambda^{-1/2}L\Lambda^{-1/2}$ is positive semidefinite and hence, by taking β to be small enough, all other eigenvalues of $I - \beta\Lambda^{-1/2}L\Lambda^{-1/2}$ can be guaranteed to lie in the interval [0,1). Since $I - \beta\Lambda^{-\frac{1}{2}}L\Lambda^{-\frac{1}{2}}$ is a symmetric matrix, the induced 2-norm coincides with the spectral radius, thus

$$\begin{aligned}
&\|(I - \Lambda^{-1}\beta L)(\boldsymbol{\lambda}_t - \lambda^t\mathbf{1})\| \\
=&\|\Lambda^{-\frac{1}{2}}(I - \beta\Lambda^{-\frac{1}{2}}L\Lambda^{-\frac{1}{2}})\Lambda^{\frac{1}{2}}(\boldsymbol{\lambda}_t - \lambda^*(t)\mathbf{1})\| \\
\leq&\|\Lambda^{-\frac{1}{2}}\|\|I - \beta\Lambda^{-\frac{1}{2}}L\Lambda^{-\frac{1}{2}}\|\|\Lambda^{\frac{1}{2}}\|\|\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}\| \\
\leq&(1-\rho)\max_{i\in\mathcal{N}} a_i^{\frac{1}{2}} \max_{i\in\mathcal{N}}(a_i)^{-\frac{1}{2}}\|\boldsymbol{\lambda}_t - \lambda^t\mathbf{1}\|,
\end{aligned} \tag{sp-14}$$

where $1 - \rho \in [0, 1)$ is the second largest eigenvalue of $I - \beta\Lambda^{-1}L$. As long as (28) holds, standard algebraic manipulations now lead to the desired conclusion. ∎