Proofs for Smart Inverter for Voltage Regulation: Physical and Market Implementation

Chenye Wu, *Member, IEEE,* Gabriela Hug, *Senior Member, IEEE,* and Soummya Kar, *Member, IEEE*

A. Proof for Equation (30)

Note that

$$\frac{\partial \lambda}{\partial p_i^g} = \sum_{k=1}^{n} 2\kappa(v_k - v_k^{ref})\frac{\partial v_k}{\partial p_i^g} = 2\kappa\sum_{k=1}^{n}(v_k - v_k^{ref})R_{ki},$$

$$\frac{\partial \lambda}{\partial q_i^g} = \sum_{k=1}^{n} 2\kappa(v_k - v_k^{ref})\frac{\partial v_k}{\partial q_i^g} = 2\kappa\sum_{k=1}^{n}(v_k - v_k^{ref})X_{ki}.$$

Hence, we have

$$\begin{aligned}
&\frac{1}{2\kappa}(\nabla\lambda(\mathbf{a}^1) - \nabla\lambda(\mathbf{a}^2))^T(\mathbf{a}^1 - \mathbf{a}^2)\\
=&\sum_{i=1}^{n}\sum_{k=1}^{n}(v_k(\mathbf{a}^1) - v_k(\mathbf{a}^2))R_{ki}(p_i^1 - p_i^2)\\
&+\sum_{i=1}^{n}\sum_{k=1}^{n}(v_k(\mathbf{a}^1) - v_k(\mathbf{a}^2))X_{ki}(q_i^1 - q_i^2)\\
=&\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}R_{kj}R_{ki}(p_j^1 - p_j^2)(p_i^1 - p_i^2)\\
&+2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}R_{kj}X_{ki}(p_j^1 - p_j^2)(q_i^1 - q_i^2)\\
&+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}X_{kj}X_{ki}(q_j^1 - q_j^2)(q_i^1 - q_i^2)\\
=&\sum_{k=1}^{n}\left(\sum_{i=1}^{n}R_{ki}(p_i^1 - p_i^2)\right)^2 + \sum_{k=1}^{n}\left(\sum_{i=1}^{n}X_{ki}(q_i^1 - q_i^2)\right)^2\\
&+2\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}R_{kj}X_{ki}(p_j^1 - p_j^2)(q_i^1 - q_i^2)\\
=&\sum_{k=1}^{n}\left(\sum_{i=1}^{n}R_{ki}(p_i^1 - p_i^2) + \sum_{i=1}^{n}X_{ki}(q_i^1 - q_i^2)\right)^2\\
\geq&\left(\sum_{k=1}^{n}\min\{\min_i R_{ki}, \min_i X_{ki}\}\right)\|\mathbf{a}^1 - \mathbf{a}^2\|_2^2,
\end{aligned}$$

which yields equation (30).

B. Proof for Theorem 2

$$\begin{aligned}
&\|\mathbf{x}^{t+1} - \mathbf{x}^*\|_2^2 = \|\mathbf{x}^t - \alpha\mathbf{g}^t - \mathbf{x}^*\|_2^2\\
=&\|\mathbf{x}^t - \mathbf{x}^*\|_2^2 - 2\alpha(\mathbf{g}^t)^T(\mathbf{x}^t - \mathbf{x}^*) + \alpha^2\|\mathbf{x}^t\|_2^2\\
\leq&\|\mathbf{x}^t - \mathbf{x}^*\|_2^2 - 2\alpha m\|\mathbf{x}^t - \mathbf{x}^*\|_2^2 + \alpha^2\|\mathbf{g}^t\|_2^2\\
\leq&(1-2\alpha m)^t\|\mathbf{x}^1 - \mathbf{x}^*\|_2^2\\
&\quad+\alpha(1-(1-2\alpha m)^t)/2m\max_{1\leq k\leq t}\|\mathbf{g}^k\|_2^2\\
\leq&(1-2\alpha m)^t\|\mathbf{x}^1 - \mathbf{x}^*\|_2^2 + \alpha/2m\max_{1\leq k\leq t}\|\mathbf{g}^k\|_2^2.
\end{aligned} \tag{36}$$

The first inequality utilizes the fact that $f(\mathbf{x})$ is strictly convex, with parameter m. The last two inequalities are standard manipulations.

C. Proof for Theorem 3

This proof aligns with that of Theorem 2.

$$\begin{aligned}
&\|\mathbf{a}^{t+1} - \mathbf{a}^*\|_2^2 = \|\mathbf{a}^t - \alpha\hat{\mathbf{g}}^t - \mathbf{a}^*\|_2^2\\
=&\|\mathbf{a}^t - \alpha(\mathbf{g}^t + \xi^t) - \mathbf{a}^*\|_2^2\\
=&\|\mathbf{a}^t - \mathbf{a}^*\|_2^2 - 2\alpha(\mathbf{g}^t + \xi^t)^T(\mathbf{a}^t - \mathbf{a}^*) + \alpha^2\|\mathbf{g}^t + \xi^t\|_2^2\\
\leq&\|\mathbf{a}^t - \mathbf{a}^*\|_2^2 - 2\alpha\gamma\|\mathbf{a}^t - \mathbf{a}^*\|_2^2\\
&\quad-2\alpha(\xi^t)^T(\mathbf{a}^t - \mathbf{a}^*) + \alpha^2\|\mathbf{g}^t + \xi^t\|_2^2\\
\leq&(1-2\gamma\alpha)^t\|\mathbf{a}^1 - \mathbf{a}^*\|_2^2\\
&\quad+(1-(1-2\gamma\alpha)^t)/\gamma\max_{1\leq k\leq t}|(\xi^k)^T(\mathbf{a}^k - \mathbf{a}^*)|\\
&\quad+\alpha(1-(1-2\gamma\alpha)^t)/2\gamma\max_{1\leq k\leq t}\|\mathbf{g}^k + \xi^k\|_2^2\\
\leq&(1-2\gamma\alpha)^t\|\mathbf{a}^1 - \mathbf{a}^*\|_2^2 + 1/\gamma\max_{1\leq k\leq t}\|\xi^k\|\|\mathbf{a}^k - \mathbf{a}^*\|\\
&\quad+\alpha/2\gamma\max_{1\leq k\leq t}\|\mathbf{g}^k + \xi^k\|_2^2.
\end{aligned} \tag{37}$$

Again, the first inequality utilizes the fact that $h(\mathbf{a})$ is strictly convex. The last inequality uses the Cauchy-Schwarz inequality. Note that, due to the strong convexity of problem (15), it admits a unique global minimizer $\mathbf{a}^*$.

For the second part, we propose the sufficient condition, under which the convergence is guaranteed. We have

$$\begin{aligned}
&\|\mathbf{a}^{t+1} - \mathbf{a}^*\|_2^2 = \|\mathbf{a}^t - \alpha\hat{\mathbf{g}}^t - \mathbf{a}^*\|_2^2\\
\leq&\|\mathbf{a}^t - \mathbf{a}^*\|_2^2 - 2\alpha\gamma\|\mathbf{a}^t - \mathbf{a}^*\|_2^2\\
&\quad-2\alpha(\xi^t)^T(\mathbf{a}^t - \mathbf{a}^*) + \alpha^2\|\mathbf{g}^t + \xi^t\|_2^2\\
\leq&\|\mathbf{a}^t - \mathbf{a}^*\|_2^2 - 2\alpha\gamma\|\mathbf{a}^t - \mathbf{a}^*\|_2^2\\
&\quad+2\alpha\|\xi^t\|\|\mathbf{a}^t - \mathbf{a}^*\| + \alpha^2\|\mathbf{g}^t + \xi^t\|_2^2\\
\leq&\|\mathbf{a}^t - \mathbf{a}^*\|_2^2 - \alpha\gamma\|\mathbf{a}^t - \mathbf{a}^*\|_2^2\\
\leq&(1-\gamma\alpha)^t\|\mathbf{a}^1 - \mathbf{a}^*\|_2^2.
\end{aligned} \tag{38}$$

In the third inequality, we use the assumption that for all $t = 1, \cdots, T$,

$$2\|\xi^t\|\|\mathbf{a}^t - \mathbf{a}^*\| + \alpha\|\mathbf{g}^t + \xi^t\|_2^2 \leq \gamma\|\mathbf{a}^1 - \mathbf{a}^*\|_2^2. \tag{39}$$

This completes the convergence proof.

D. Proof for Lemma 5

It suffices to verify

$$u_i(a_i^\dagger, a_{-i}) - u_i(a_i^\ddagger, a_{-i}) = \Psi(a_i^\dagger, a_{-i}) - \Psi(a_i^\ddagger, a_{-i}). \tag{40}$$

The lemma immediate follows.

E. Proof for Theorem 6

This is one of the nice properties of potential games [20]. Intuitively, at the global minimizers to problem (9), every player's profit has automatically been maximized due to the potential function Ψ.